Embedding of Right-Angled Artin and Coxeter Groups into Products of Trees

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Abstract

A finitely generated group Γ is a right-angled Artin group if the only relations in Γ are commutators of generators. A finitely generated group Γ is a right-angled Coxeter group if all generators have order two and the other relations are commutators of generators.
The aim of this thesis is to show that these kinds of groups admit a quasi-isometric embedding into products of binary trees. An important aspect is that the number of trees is bounded by the coloring number of the group, which is the minimal number of colors we need to color the generators such that commuting generators have different colors. The basic idea of the embedding is a vector valued diary map with an aperiodic decoration.

Zusammenfassung

Eine endlich erzeugte Gruppe Γ ist eine rechtwinklige Artingruppe, falls alle Relationen in Γ Kommutatoren der Erzeugenden sind. Eine endlich erzeugte Gruppe Γ ist eine rechtwinklige Coxetergruppe, falls alle Erzeugenden die Ordnung zwei haben und die anderen Relationen Kommutatoren der Erzeugenden sind.
In dieser Arbeit wird gezeigt, dass diese beiden Arten von Gruppen quasiisometrisch in Produkte von binären Bäumen eingebettet werden können. Dabei ist die Anzahl der benötigten Bäume durch die chromatische Zahl der Gruppe gegeben, welche die minimal Anzahl der Farben ist, die man benötigt, um die Erzeugenden so zu färben, dass kommutierende Erzeugende verschiedene Farben haben. Die grundlegende Idee der Einbettung ist eine vektorwertige Tagebuchabbildung mit einer aperiodischen Dekorierung.

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Chapter 1

Introduction

A finitely generated group Γ is a *right-angled Artin group* if the only relations in Γ are commutators of generators. A finitely generated group Γ is a *right-angled Coxeter group* if all generators have order two and the other relations are commutators of generators.

A group Γ is called *n-colored* if the chromatic number of Γ is n, i.e. if we need at least n colors to color the elements of a generator set such that commuting elements have different colors.

The Cayley graph $\mathcal{C}(\Gamma, S)$ equipped with the path metric is unique up to quasi-isometry depending on the generator system. Hence quasi-isometric embeddings are the natural kind of maps for studying groups from a geometric point of view.

The initial question of this thesis was whether finitely generated right-angled Coxeter groups admit a quasi-isometric embedding into finite products of binary trees. Here we consider the path metric on trees and the l_1-metric on products of trees.

The answer is positive and the result can even be proven for an additional type of groups, namely right-angled Artin groups:

Theorem 1.1. *Let Γ be a finitely generated n-colored right-angled Artin or Coxeter group. Then there exists a quasi-isometric embedding $\Gamma \to T \times \cdots \times T$ of it into the product of n binary trees T.*

Products of binary trees

The range of the embedding – a product of binary trees – is the simplest possible embedding space for right-angled Artin and Coxeter groups. Indeed they are in general neither embeddible in a standard hyperbolic space H^N nor in a Hilbert space. The first follows directly from the fact that geodesic metric spaces which are embeddible in a hyperbolic geodesic metric space are hyperbolic (see e.g. [BH99], p. 402). The second was shown e.g. by Brodskiy and Sonkin in [BS08] based on results of Bourgain [Bou85] and Linial and Saks [LS03].

The number of trees in the product is invariant under quasi-isometries in the following sense:

If $\prod^n T$ is quasi-isometric to $\prod^m T$, then $m = n$.
This can be shown by considering the hyperbolic dimension. The hyperbolic dimension is a quasi-isometry invariant introduced by Buyalo and Schroeder in [BS07b] as a version of Gromov's asymptotic dimension. They prove that the hyperbolic dimension of the hyperbolic space H^n is n, $\operatorname{hypdim} H^n = n$, and that the hyperbolic dimension of a product of binary trees $\prod^n T$ is less than or equal to n, $\operatorname{hypdim} \prod^n T \leq n$.

In [BDS07] it is proven that H^n can be quasi-isometrically embedded into a product of n binary trees. Therefore:

$$\operatorname{hypdim} H^n \leq \operatorname{hypdim} \prod^n T.$$

Combined with the above, we obtain $\operatorname{hypdim} \prod^n T = n$. Since hyperbolic dimension is a quasi-isometry invariant, $\prod^n T$ is not quasi-isometric to $\prod^m T$ for $n \neq m$.

Another remarkable fact is that the n-fold product of binary trees can itself be interpreted as the Cayley graph of an n-colored right-angled Coxeter group. Indeed, the Cayley graph of the right-angled Coxeter group

$$W = \left\langle a_1, a_2, a_3 \mid a_i^2 = 1 \right\rangle$$

is a binary tree, products of right-angled Coxeter groups are right-angled Coxeter groups, thus $\prod^n T$ is the Cayley graph of the n-colored right-angled Coxeter group $\prod^n W$.

In the case of Artin groups, consider the right-angled Artin group

$$A = \langle b_1, b_2 \mid \emptyset \rangle ,$$

which is the free group generated by $\{b_1, b_2\}$. Its Cayley graph is a tree with vertices of valence 4, therefore it is quasi-isometric to the binary tree T. This implies that the n-colored right-angled Artin group $\prod^n A$ is quasi-isometric to $\prod^n T$.

Previous results

Previous work by Dranishnikov and Schroeder [DS04] proved that the Cayley graph of a finitely generated n-colored right-angled Coxeter group admits a bi-Lipschitz embedding into a product of n locally finite trees.

In [DS05] same authors improved this result for hyperbolic right-angled Coxeter groups by constructing an embedding into a product of n binary trees, which are trees of uniformly bounded valence. The fundamental idea is the diary concept in which an aperiodic decoration based on the Morse-Thue sequence plays an important role.

One approach to generalize this result is to consider other hyperbolic groups. This is done in [BDS07] by Buyalo, Dranishnikov and Schroeder using and upgrading the idea of the diary map. The exact statement is: Every hyperbolic group Γ admits a quasi-isometric embedding into the product of $m+1$ binary trees, where $m = \dim \partial_\infty \Gamma$ is the topological dimension of the boundary at infinity.

Another approach is to investigate whether the condition of hyperbolicity is really necessary for an embedding into products of binary trees. Adopting the concepts and modifying the embedding we were able to prove the following result for 2-colored right-angled Coxeter groups, which is reported in [Rul07].

Theorem 1.2. *Let Γ be a finitely generated 2-colored right-angled Coxeter group and T be a binary tree. Then there exists a quasi-isometric embedding $\Gamma \to T \times T$ of it into the product of two binary trees.*

Instead of a common decoration for all generators we attach its own aperiodic sequence to each generator. This allows us to collect more information in diary entries, thus the embedding becomes quasi-isometric for 2-colored right-angled Coxeter groups. Unfortunately, this construction was not powerful enough to also work for higher chromatic numbers. In this case, more profound changes are required.

This thesis

This thesis describes an alteration of the diary map to generalize the previous work [DS05, Rul07] and prove the main result, Theorem 1.1.

The idea of the modification of the diary map, which is used to construct the embedding, is to build more diaries, i.e. generate not a diary for the whole word, but for certain subwords. We obtain more information about a group element since these subwords are less dependent of the canonical form. By the construction these several diaries can be arranged into a vector such that we obtain a vector valued diary. Thus we still need not more than n (coloring number) trees for the image space.

The thesis is structured as follows. Chapter 2 describes basic notions, mainly Artin and Coxeter groups, word representations for group elements, and the Morse-Thue sequence. In Chapter 3 several maps with their properties are explained, primarily the diary map and the decoration map. Moreover, the method to combine these maps into the embedding $\boldsymbol{E}$ is introduced, followed by an example and the explanation of the necessity of two of the maps. Eventually the main theorem is proven in Chapter 4.

Open questions

Q1: Is there a quasi-isometric embedding of arbitrary Coxeter groups into finite products of binary trees?

We succeeded in generalizing the embedding result [DS05] of hyperbolic right-angled Coxeter groups to right-angled Coxeter groups. The natural question is now, whether further generalizations are possible.

Otherwise, since the embedding can be constructed also for right-angled Artin groups, it is reasonable to suppose that the "right-angled property" plays an important role. We essentially use the combinatorial property that two reduced words to the same group element can be transformed in each other by transposing of commuting letters.

Another question is about preserving the group action. The diary construction absolutely destroys the group structure. Consequently, our embedding is not equivariant.

Q2: Which kinds of groups admit an equivariant embedding into finite products of binary trees?

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Chapter 2

Preliminaries

In this chapter the required notions, definitions, and some results are stated.

Basically, the facts are of common knowledge and can be found in [BH99, BS07a, Cha07, Dav08, dlH00]. The last section is on the Morse-Thue sequence and, as its application, on a square free sequence, which were introduced in [Mor21, Thu06, Thu12].

2.1 Notation, conventions, and definitions

As usually, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ the set of nonnegative integers, and $\mathbb{N}_{>0}$ the set of positive integers. If A is a set, $\#A$ is its cardinality and $\mathcal{P}(A)$ its power set.

We only consider finitely generated groups. If not stated otherwise, Γ denotes a finitely generated n-colored right-angled Artin or Coxeter group with a generator system S and colors $c_1, \ldots, c_n$. An element from Γ is usually called γ, a color is usually called a. (For precise definitions see Section 2.3.)

The declaration of an index set is omitted if it is obvious, e.g. we say "w_i" for "w_i, $i \in \{1, \ldots, p+1\}$" if it is clear, that i can only have values 1 to $p+1$.

Since the thesis intends to find a quasi-isometric embedding, the first definition states what it is.

Definition 2.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is a* quasi-isometric embedding *if there are $a \geq 1$, $b \geq 0$ such that for all $x, \overline{x} \in X$*

$$\frac{1}{a} d_X(x, \overline{x}) - b \leq d_Y(f(x), f(\overline{x})) \leq a d_X(x, \overline{x}) + b.$$

It is a quasi-isometry *if additionally there is a $D \geq 0$ such that every element $y \in Y$ lies within distance D from $f(X)$. Then the spaces X and Y are called* quasi-isometric*.*

Remark 2.2. *In order to prove that a map f is a quasi-isometric embedding, it suffices to show independently the two inequalities, i.e. there are $a, c > 0$, $b, d \geq 0$ such that for all $x, \overline{x} \in X$ it holds:*

$$\frac{1}{a} d_X(x, \overline{x}) - b \leq d_Y(f(x), f(\overline{x})) \leq c d_X(x, \overline{x}) + d.$$

We will show the main result in this way.

2.2 Words

Words over alphabets

We start with the standard notion of an *alphabet* A as a (finite or infinite) nonempty set. The elements of A are called *letters* (or *characters*).

A *word* (or *string*) w over an alphabet A is a finite sequence of characters x_i from A:

$$w = x_1 \dots x_k.$$

We call the number of characters in w the *length* $\ell(w)$ *of* w:

$$\ell(w) = k.$$

The unique word with the length 0 is denoted by ε and called the *empty word*.

For any alphabet A the set of all words over A is denoted by A^* and the set of all nonempty words by A^+:

$$A^* = \{x_1 \dots x_k \mid x_i \in A, k \in \mathbb{N}\}, \quad A^+ = A^* \backslash \{\varepsilon\}.$$

Two words $u = y_1 \dots y_p$ and $v = z_1 \dots z_q$ in A^* can be *concatenated* to a word $uv = y_1 \dots y_p z_1 \dots z_q \in A^*$. The length is compatible with this operation: $\ell(uv) = \ell(u) + \ell(v)$.

Let w be a word. It is useful to consider w *to the power of* n, which is a n-fold repetition of w:

$$\begin{aligned} w^0 &= \varepsilon, \\ w^i &= w^{i-1} w \quad \text{for} \quad i \in \mathbb{N}_{>0}. \end{aligned}$$

Word representation for group elements

Let Γ be a group with a generator system S. Every group element $\gamma \in \Gamma$ has a representation as a product of generators and their inverses: $\gamma = s_1 \cdot s_2 \cdot \ldots \cdot s_k$, which can be interpreted as a word in $(S \cup S^{-1})^*$. Conversely, every word w

in $(S \cup S^{-1})^*$ corresponds to an element $\gamma \in \Gamma$. Denote this natural projection by π:

$$\begin{aligned} \pi: \quad (S \cup S^{-1})^* &\to \Gamma \\ s_1 s_2 \dots s_k &\mapsto s_1 \cdot s_2 \cdot \dots \cdot s_k. \end{aligned}$$

The image of the empty word ε is the identity element 1 of the group:

$$\pi(\varepsilon) = 1.$$

Let $w = x_1 \dots x_k \in (S \cup S^{-1})^*$ be a representation of an element γ in Γ, then we define the *inverse word* w^{-1} by:

$$w^{-1} = x_k^{-1} \dots x_1^{-1},$$

which is a representation of the inverse element γ^{-1} of γ:

$$\pi(w) = \gamma \quad \Rightarrow \quad \pi(w^{-1}) = \gamma^{-1}.$$

We will casually identify words with corresponding group elements.

The map π is obviously surjective. Thus we can define the *length* (or *norm*) $\ell(\gamma)$ of a group element $\gamma \in \Gamma$ using the notion of the word length:

$$\ell(\gamma) = \min\{\ell(w) \mid \pi(w) = \gamma\}.$$

In this way we obtain the *word metric* on Γ:

$$d(\gamma, \overline{\gamma}) = \ell(\gamma^{-1}\overline{\gamma}).$$

Consequently the length of an element $\gamma \in \Gamma$ can be expressed in terms of the word metric: $\ell(\gamma) = d(\gamma, 1)$.

A word w with $\ell(w) = \ell(\pi(w))$ is said to be *reduced*.

The map π is not injective, not even its restriction to the reduced words. Nevertheless it is possible to find a unique word $w \in \pi^{-1}(\gamma)$ to every $\gamma \in \Gamma$. We will give a canonical representation in Section 3.1.

Note that the word metric (and hence the length) depends on the generator system S. However, the word metric is unique up to quasi-isometry.

Word trees

We define a tree structure on words of an alphabet A. Thus let A^* be the set of vertices. Two words are connected by an edge if one of them emanates from the other by concatenating a letter from A. More precisely, the edges are of the form (w, wa) with $w \in A^*$ and $a \in A$.

Denote such a tree by T_A and consider the path metric on it, i.e. the distance between two vertices v and w is the smallest number of edges in a path connecting them. The distance between a vertex w and the root vertex, the empty word ε, will be denoted by $|w|$, the distance between two vertices v, w by $|vw|$. Note that $|w| = \ell(w)$.

On a product of trees we assume a sum metric (l_1-metric), i.e. the distance in the product is the sum of the distances in the factors. The notations are analogous to above $|\cdot|_1$ and $|\cdot\cdot|_1$.

Let $2 \leq \#A < \infty$. Elements of A can be encoded as binary strings of length not greater than $\log_2(\#A) + 1$. Therefore, T_A is quasi-isometric to a tree T_Ω with $\#\Omega = 2$, which is a subtree of the binary tree T (every vertex has valence 3). And thus:

Remark 2.3. *If the alphabet A is finite, then T_A can be quasi-isometrically embedded into the binary tree.*

2.3 Artin and Coxeter groups

Considering its Cayley graph, a group can be seen as a metric space.

Cayley graph

Every group Γ with a generator system S can be completely described by its *Cayley graph* $\mathcal{C}(\Gamma, S)$. The vertex set is Γ, and for all $s \in S$ there is an edge labeled by s connecting γ to γs.

Considering the path metric on the Cayley graph, we again obtain the word metric on Γ. The Cayley graph is even a geodesic metric space, but in general not uniquely geodesic. Here, a geodesic from γ to $\overline{\gamma}$ is defined as a path $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k = \overline{\gamma}$ with $d(\gamma_i, \gamma_j) = 1$ and $d(\gamma, \overline{\gamma}) = k$.

Remark 2.4. *There is a natural action from the left of the group Γ on its Cayley graph $\mathcal{C}(\Gamma, S)$, which is transitive on the set of vertices, and the word metric is Γ-invariant.*

Every path from γ to $\overline{\gamma}$ in $\mathcal{C}(\Gamma, S)$ corresponds to a word $w \in (S \cup S^{-1})^*$ and the reverse path from $\overline{\gamma}$ to γ corresponds to w^{-1}, such that $\gamma \cdot \pi(w) = \overline{\gamma}$ and $\overline{\gamma} \cdot \pi(w^{-1}) = \gamma$. It is easy to see that the paths from γ to $\overline{\gamma}$ correspond exactly to $\pi^{-1}(\gamma^{-1}\overline{\gamma})$ and that geodesics correspond to reduced words.

Definition of Artin and Coxeter groups

For the definition of Artin and Coxeter groups, consider first a Coxeter matrix. Let $S = \{s_1, \ldots, s_k\}$ be a finite set. A *Coxeter matrix* $M = (m_{ij})$ is a $k \times k$ symmetric matrix with $m_{ij} \in \mathbb{N}_{>0} \cup \{\infty\}$, where $m_{ij} = 1$ if $i = j$ and $m_{ij} > 1$ if $i \neq j$.

Now we can define a *Coxeter group* W as a group with the generator set S and relations of the form $(s_i s_j)^{m_{ij}} = 1$:

$$W = \langle s_1, \ldots, s_k \mid (s_i s_j)^{m_{ij}} = 1 \rangle.$$

By $(s_i s_j)^\infty = 1$ we mean that there is no relation imposed between s_i and s_j. Furthermore $m_{ii} = 1$ means that all the generators have order two and thus $s_i = s_i^{-1}$. And if $m_{ij} = 2$, then s_i and s_j commute, since:

$$1 = (s_i s_j)^2 = s_i s_j s_i s_j \quad \Leftrightarrow \quad s_j = s_i s_j s_i \quad \Leftrightarrow \quad s_j s_i = s_i s_j.$$

For the definition of an Artin group we again use a Coxeter matrix M. An *Artin group* A with the generator system S is defined as following:

$$A = \langle s_1, \ldots, s_k \mid \underbrace{s_i s_j s_i \ldots}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}} \rangle.$$

As in the case of Coxeter groups, $m_{ij} = \infty$ means that there is no relation between s_i and s_j, and $m_{ij} = 2$ means that s_i and s_j commute.

(The expression $\underbrace{s_i s_j s_i \ldots}_{m_{ij}}$ stands for the word $s_i s_j s_i \ldots$ with m_{ij} letters (not pairs). If m_{ij} is even, the word ends with s_j, if m_{ij} is odd, then the last letter is s_i.)

Remark 2.5. *If we add the relations $s_i^2 = 1$ to an Artin group A, we obtain the Coxeter group W defined by the same Coxeter matrix:*

$$\begin{aligned} W &= \langle s_1, \ldots, s_k \mid (s_i s_i)^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle \\ &= \langle s_1, \ldots, s_k \mid (s_i s_i)^2 = 1, \underbrace{s_i s_j s_i \ldots}_{m_{ij}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}} \rangle. \end{aligned}$$

An Artin or a Coxeter group Γ is called *right-angled* if the m_{ij} are in $\{1, 2, \infty\}$. In this case every word can be shortened to a reduced word by a sequence of the following operations:

1. Delete a subword of the form $s_i s_i^{-1}$ or $s_i^{-1} s_i$.
2. Replace a subword $s_i^{\pm 1} s_j^{\pm 1}$ by $s_j^{\pm 1} s_i^{\pm 1}$ if $m_{ij} = 2$.

Proposition 2.6. *Let Γ be an Artin or a Coxeter group. Two reduced words w_1, w_2 belonging to the same element $\pi(w_1) = \pi(w_2) \in \Gamma$ can be transformed one into the other by using only the second operation: transposing commuting generators.*

Proof: For Coxeter groups this statement was originally proven by Tits and reproven by Davis in [Dav08], Theorem 3.4.2. A more general version including Artin groups is proven by Green in [Gre90]. □

Although right-angled Artin or Coxeter groups are in general not hyperbolic, there is a 0-thin triangle between any three points, i.e. any three vertices span a geodesic tripod. This property is essential for the proof of the main theorem. The following lemma was formulated and proven in [DS05] for Coxeter groups. It works analogous also for Artin groups.

Lemma 2.7. *Let Γ be a right-angled Artin or Coxeter group and $\alpha, \beta, \gamma \in \Gamma$, then there is a $\delta \in \Gamma$ such that:*

$$\begin{aligned} d(\alpha, \delta) + d(\delta, \beta) &= d(\alpha, \beta), \\ d(\beta, \delta) + d(\delta, \gamma) &= d(\beta, \gamma), \\ d(\alpha, \delta) + d(\delta, \gamma) &= d(\alpha, \gamma). \end{aligned}$$

In other words, the concatenation of geodesics between δ and two of the points α, β, and γ is a geodesic between these points.

Proof: It suffices to consider the case $\gamma = 1$ by reason of Γ-invariance (Remark 2.4). We assume $\alpha, \beta \in \Gamma$, and $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_k = \beta$ a geodesic in between. Denote the length of γ_i by l_i and choose a reduced word w_i for every γ_i.

Since there are $t_i \in S \cup S^{-1}$ such that $\pi(w_i t_i) = \gamma_{i+1}$ and the relations of Γ have even lengths, it holds that two vertices connected by an edge have different lengths $l_i \neq l_{i+1}$.

If there is a local maximum of the lengths at i, i.e. $l_{i-1} < l_i > l_{i+1}$, then there are $s, t \in S$ such that $w_{i-1}s, w_{i+1}t \in \pi^{-1}(\gamma_i)$ are reduced representations of γ_i. Because these representations can be transformed one into the other by transposition of adjacent letters (Proposition 2.6), $st = ts$ holds and there is a word w such that $wt \in \pi^{-1}(\gamma_{i-1})$ and $ws \in \pi^{-1}(\gamma_{i+1})$ are reduced words of γ_{i-1} and γ_{i+1} respectively. We replace γ_i in the geodesic by $\overline{\gamma}_i := \pi(w)$ and obtain a new geodesic with $l_{i-1} > \overline{l}_i < l_{i+1}$.

Finitely many repetitions of this step provide a geodesic without local maxima except at the endpoints. On the other hand, two vertices connected by an edge have different lengths, and therefore there is a global minimum of lengths at i_{min}. The element $\delta := \gamma_{i_{min}}$ satisfies the requirements, because:

$$\begin{aligned} d(\alpha, 1) &= \ell_0 = i_{min} + l_{i_{min}} = d(\alpha, \gamma_{i_{min}}) + d(\gamma_{i_{min}}, 1) \\ d(1, \beta) &= \ell_k = l_{i_{min}} + (l_k - i_{min}) = d(1, \gamma_{i_{min}}) + d(\gamma_{i_{min}}, \beta). \end{aligned}$$

□

Colorings

A right-angled Artin or Coxeter group Γ can be completely described by a graph $N(\Gamma, S)$. The vertices of this graph are the generators S of Γ. The generators s_i and s_j are connected by an edge if and only if they commute.

Remark 2.8. *For a Coxeter group Γ the defining graph $N(\Gamma, S)$ is the one-skeleton of the nerve in the Davis complex of Γ.*

The *chromatic number* of a graph is the minimal number of colors needed to color the vertices such that two adjacent vertices have different colors. We define the *chromatic number* of a group Γ by the minimal chromatic number of the graph $N(\Gamma, S)$ depending on the generator system S. We also call Γ *n-colored* if its chromatic number is n.

We consider a minimal coloring of the generators by colors $c_1, \dots, c_n$. It induces a disjoint decomposition of the generator set $S = \bigcup_{\lambda=1}^{n} S_{c_\lambda}$ into subsets S_{c_λ} colored by c_λ. Note that two elements of the same color do not commute. To simplify the notation we use letters $a, b, c, \dots$ for the colors and denote the generators of color a by $a_1, a_2, \dots$. If a generator s has color a then s^{-1} also has color a.

We denote by $\ell_a(\gamma)$ the number of letters of color a in a reduced representation of γ. Obviously, the following holds: $\ell_a(\gamma) + \ell_b(\gamma) + \ell_c(\gamma) + \dots = \ell(\gamma)$.

2.4 The square free sequence

In this chapter we introduce the square free sequence which is used for the decoration map in Section 3.3.

Definition 2.9. *Let A be an alphabet. A sequence $(a_i)_{i \in \mathbb{N}}$ with values in A is called* square free *if it does not contain a substring of the form ww for any nonempty string w over the alphabet A.*

In 1906 Axel Thue proved the existence of sequences of this kind in [Thu06]. In a later and more detailed paper [Thu12] he found a different method to construct a square free sequence, namely out of an overlap free sequence. Moreover, he gave an example of an overlap free sequence, which nowadays is named after him and Marston Morse, who rediscovered and studied it later in [Mor21].

Definition 2.10. *Beginning with 0 we substitute 0 by 01 and 1 by 10:*

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

The limit sequence of this substitution is the Morse-Thue sequence $(m_i)_{i\in\mathbb{N}}$.

From this definition we obtain an *inductive description* of the sequence:

$$m(0) = 0 \text{ and for all } i \in \mathbb{N}: \; m(2i) = m(i), \; m(2i+1) = 1 - m(2i).$$

Proposition 2.11. *The Morse-Thue sequence is overlap free, i.e. there is no substring of the form $ababa$ in the sequence where a and b are strings over $\{0,1\}$ and a is not empty.*

Proof: Satz 6 in [Thu12]. □

The method Thue uses to construct a *square free sequence* from an overlap free sequence is the following substitution:

$$011 \to 2, \quad 01 \to 1, \quad 0 \to 0.$$

In simple terms this means counting the numbers of 1s between 0s.

In this way we obtain the sequence:

$$(q_i)_{i\in\mathbb{N}} = 21020121012\dots$$

from the Morse-Thue sequence, i.e. q_i is the number of 1s between the i-th and (i+1)-th occurrence of 0 in $(m_i)_{i\in\mathbb{N}}$.

It can be seen that this sequence is square free as follows: Assume there exists a substring ww in $(q_i)_{i\in\mathbb{N}}$ with $w = x_1 \dots x_n$, $n > 0$. Let $a = 0$ and $b = 1^{x_1}01^{x_2}0\dots01^{x_n}$ (with $1^0 = \varepsilon$, $1^1 = 1$, and $1^2 = 11$). Then the string $ababa$ is a substring of the Morse-Thue sequence in contradiction to the overlap free property.

Chapter 3

Definition of the embedding

Let Γ be a finitely generated n-colored right-angled Artin or Coxeter group with a generator system S. In this chapter the map $\boldsymbol{E}$ from Γ into an n-fold product of trees will be introduced. The proof that $\boldsymbol{E}$ is a quasi-isometric embedding will be given in Chapter 4.

The structure of this chapter is the following. In Sections 3.1 to 3.4 we introduce four maps which in 3.5 will be combined to the desired embedding. In Sections 3.6 and 3.7 the necessity of the decoration map (Section 3.3) and of the restriction map (Section 3.2) will be explained on the basis of counter-examples.

3.1 Canonical representation ν_{c_λ}

In this section we will assign a unique word $w \in \pi^{-1}(\gamma)$ to every $\gamma \in \Gamma$ such that all the letters of a chosen color are as far left as possible. Before describing the canonical representation, we need some preparations.

Moving one single letter as far left as possible is well defined in the following sense:

Lemma 3.1. *Let γ be in Γ with a reduced representation $x_1 \dots x_m$ where $x_i \in S \cup S^{-1}$. For every i there is a unique decomposition $\gamma = \alpha_1 x_i \alpha_2$ with $\alpha_1, \alpha_2 \in \Gamma$ and $\ell(\alpha_1)$ minimal.*

We remark that α_1 is unique as a group element, it does not necessarily have a unique word representation.

Proof: Since there are only finitely many reduced representations of γ, we can take one of them with the minimal number of letters left from x_i. Denote the corresponding decomposition by $\gamma = \alpha_1 x_i \alpha_2$. Now we have to prove the uniqueness.

Let $\alpha_1' x_i \alpha_2'$ be a decomposition of γ with $\ell(\alpha_1') = \ell(\alpha_1)$. Then (the word which defines) $\alpha_1 x_i \alpha_2$ can be transformed into (a word belonging to) $\alpha_1' x_i \alpha_2'$ only by transposing adjacent letters.

The length of the subword to the left of x_i is only affected by moving letters from α_1 to the right of x_i and moving letters from α_2 to the left of x_i. Because all of the generators of the first type can be moved to the right of the generators of the second type, they commute. Thus the order of these movements is not relevant, and we can assume that the transpositions of the first type take place first.

However, we cannot move any generator to the right of x_i because of the minimal length of α_1. Any transposition of the second type makes α_1' longer than α_1 and this contradicts the assumption.

We can conclude that no transpositions of other letters with x_i are possible. This means that α_1' can be achieved from α_1 by transposing commuting generators, and therefore $\alpha_1' = \alpha_1$. □

Remark 3.2. *The unique decomposition from Lemma 3.1 can be also obtained in a constructive way, namely by moving subsequently every x_j for j from $i-1$ to 1 as far as possible to the right by transposition of commuting letters.*

Since two generators of the same color do not commute, we have:

Remark 3.3. *Let γ be in Γ, let c_λ be a color. The order of generators of color c_λ (i.e. $S_{c_\lambda} \cup S_{c_\lambda}^{-1}$) in γ is uniquely determined.*

Canonical representation

Let $c_1 < c_2 < \cdots < c_n$ be an ordering on the colors. Let a be an arbitrary color, e.g. $a = c_\lambda$. We define the *(canonical) a-representation* $\nu_a(\gamma)$ of a $\gamma \in \Gamma$ in the following way:

Step 1: The order of letters of color a is unique by Remark 3.3, we denote them by $a_1, \ldots, a_p$ with $p := \ell_a(\gamma)$.

1. Applying Lemma 3.1 with respect to a_1, we obtain the decomposition:
$$\gamma_1 a_1 \alpha_2.$$
2. For $2 \leq i \leq p$ decompose α_i with respect to a_i to obtain:
$$\gamma_i a_i \alpha_{i+1}.$$

Altogether, γ is uniquely decomposed as follows:
$$\gamma = \gamma_1 a_1 \gamma_2 a_2 \ldots a_m \alpha_{m+1}.$$

Step 2: Note that the elements γ_i, α_{m+1} between the a_i are generated by $S \setminus S_a$. We can apply Step 1 with respect to color c_1 to them. By subsequently decomposing the obtained words in between with respect to colors $c_2, \dots, c_{\lambda-1}, c_{\lambda+1}, \dots, c_n$, we get a decomposition, which is associated to a unique word w. Thus w can be seen as the canonical a-representation $\nu_a(\gamma)$ of the element γ.

Consequently, we can define the canonical a-representation map $\nu_a(\gamma)$:

$$\begin{aligned} \nu_a : \ \Gamma \ &\to (S \cup S^{-1})^* \\ \gamma \ &\mapsto \nu_a(\gamma). \end{aligned}$$

We will use the notation:

$$\nu_a(\gamma) = w_1 a_1 w_2 a_2 \dots a_p w_{p+1},$$

where $a_i \in S_a \cup S_a^{-1}$ and $w_i \in ((S \setminus S_a) \cup (S \setminus S_a)^{-1})^*$.

Remark 3.4. *This normal form depends on the chosen ordering of colors.*

Canonical form of products

We want to investigate how the normal form interacts with a product of group elements.

Lemma 3.5. *Let α, β be in Γ with a-representations u and v respectively:*

$$\begin{gathered} \nu_a(\alpha) = u = u_1 a_1 u_2 \dots u_p a_p u_{p+1}, \\ \nu_a(\beta) = v = v_{p+1} a_{p+1} v_{p+2} \dots v_q a_q v_{q+1}. \end{gathered}$$

Suppose uv to be reduced. Let w be the a-representation of $\alpha\beta$, i.e. $w = \nu_a(\alpha\beta)$. Then there are $w_{p+1}, \dots, w_{q+1} \in (S \cup S^{-1})^*$, such that:

$$w = u_1 a_1 \dots u_p a_p w_{p+1} a_{p+1} \dots a_q w_{q+1}.$$

Proof: The word uv is a reduced representation of $\alpha\beta$. We apply the a-representation construction to uv.

Since u is in the a-representation, u_i, $i \leq p$, are already of minimal length and in the proper order, thus the subword $u_1 a_1 \dots u_p a_p$ is not affected by the normalization. Letters $a_{p+1}, \dots, a_q$ occur in the same order. Hence with suitable $w_{p+1}, \dots, w_{q+1} \in (S \cup S^{-1})^*$ the a-representation of $\alpha\beta$ has the desired form. □

3.2 Restriction map ρ_{c_λ}

The restriction map ρ will be defined on words over an alphabet A. It reduces a word to a subword, which only consists of letters of a certain subset of A, in other words, it deletes letters that are not in this subset.

Definition 3.6. *Let A be an alphabet, the* restriction map $\rho : A^* \times \mathcal{P}(A) \to A^*$ *is defined as follows:*

- *For a letter $a \in A$,*

$$\rho(a, M) = \begin{cases} a & \text{for } a \in M, \\ \varepsilon & \text{for } a \notin M. \end{cases}$$

- *For a word $w = x_1 x_2 \dots x_k \in A^*$,*

$$\rho(w, M) = \rho(x_1, M)\rho(x_2, M) \dots \rho(x_k, M).$$

For short $\rho(w, M)$ will be written as $[w]^M$.

Notice that the elements in the image of ρ are words with no relations. They do not represent group elements.

Remark 3.7. *It follows immediately from the definition that:*

$$[v]^M [w]^M = [vw]^M$$

for $M \in \mathcal{P}(A)$, $v, w \in A^*$.

Example 3.8. $[bcabcbcbabcab]^{\{a,b\}} = babbbabab$.

Lemma 3.9. *Let Γ be a right-angled Artin or Coxeter group. Let $u = u_1 a_1 \dots a_p u_{p+1}$ and $v = v_{p+1} a_{p+1} \dots a_q v_{q+1}$ be two words in the a-canonical form and suppose uv to be reduced. Let the word $w = w_1 a_1 \dots a_q w_{q+1}$ be the a-representation of the product $\pi(u)\pi(v)$. Let be $v_{p+1} = x_k \dots x_1$.*

(i) *If one of the elements $x_k, \dots, x_1, a_{p+1}$ does not commute with $s \in S$ then*

$$[w_{p+1}]^{\{s\}} = [u_{p+1}]^{\{s\}} [v_{p+1}]^{\{s\}}.$$

(ii) *If $x_i = s$ then $[w_{p+1}]^{\{s\}} = [u_{p+1}]^{\{s\}} [v_{p+1}]^{\{s\}}$.*

(iii) *If $x_j = t$ and $x_i = s$ for $j > i$ then*

$$[w_{p+1}]^{\{s,t\}} = [\nu_a(u_{p+1} x_k \dots x_{i+1})]^{\{s,t\}} s [x_{i-1} \dots x_1]^{\{s,t\}}.$$

Proof: In Lemma 3.5 it is proven that w begins with $u_1 a_1 \dots u_p a_p$. Thus w_{p+1} consists of the letters of $u_{p+1}v_{p+1}$ without

(a) those which can be moved to the right of a_{p+1} and

(b) vanishing pairs $r^{\pm 1} r^{\mp 1}$ with $r \in S$.

Case (b) cannot occur because the word uv is reduced. The elements $x_k, \dots, x_1$ cannot be moved to the right of a_{p+1}, because v is in the a-representation. Therefore letters of type (a) are from u_{p+1}.

If s does not commute with one of the elements of v_{p+1} or with a_{p+1}, it cannot be moved to the right of these elements and thus to the right of a_{p+1}. Therefore the word $[w_{p+1}]^{\{s\}}$ consists of the letters s out of $u_{p+1}v_{p+1}$.

Additionally, these letters cannot be permuted by normalization. We conclude that the order of letters s does not change. Thus statement (i) is proven.

If $x_i = s$, then one of the letters $x_{i-1}, \dots, x_1, a_{p+1}$ does not commute with s, since v is in the a-representation. Thus (ii) follows by (i).

By (ii) we know that the number of letters s and t in w_{p+1} and in $u_{p+1}v_{p+1}$ is the same. Merely the order can be different. We show that the order in the last letters is the same. Because v is already in the a-representation, the order of letters s and t in v_{p+1} cannot be changed by normalization. However letters s and t from u_{p+1} can be moved in between. But letters t cannot pass $x_j = t$ and therefore also not $x_i = s$. Letters s can as well not pass $x_i = s$. Thus letters s and t from u_{p+1} cannot be moved in between the word $s x_{i-1} \dots x_1$ and hence the last $\ell(s[x_{i-1} \dots x_1]^{\{s,t\}})$ letters of $[w_{p+1}]^{\{s,t\}}$ are exactly $s[x_{i-1} \dots x_1]^{\{s,t\}}$. $\square$

Applying the map ρ we lose information about a word because we delete a part of the letters. The density of information about the remaining letters increases rather thereby. Now we define a map which restricts a word to several subsets of the generators.

Let $M_\lambda = \{M \subset (S \setminus S_{c_\lambda}) \cup (S \setminus S_{c_\lambda})^{-1} \mid \#M = 1, 2\}$ be the set of subsets of the generators with one or two elements, which have a different color than c_λ, and let its cardinality be $m_\lambda = \#M_\lambda$. Then define:

$$\begin{aligned} \boldsymbol{\rho_{c_\lambda}} : \quad A^* &\to (A^*)^{m_\lambda} \\ v &\mapsto \left(\rho(v, M \cup S_{c_\lambda} \cup S_{c_\lambda}^{-1})\right)_{M \in M_\lambda}. \end{aligned}$$

We intend to apply this map $\boldsymbol{\rho_{c_\lambda}}$ to the image of the map ν_{c_λ}. Thus $\boldsymbol{\rho_a}(\nu_a(\gamma))$ is a tuple of restrictions of the a-representation of γ to the letters of color a and one or two other letters.

3.3 Square free decoration $\boldsymbol{\delta}$

To counteract the periodicity of some words (cp. Section 3.6), we define an aperiodic decoration map δ, which attaches the beginning of the square free sequence $(q_i)_{i\in\mathbb{N}}$ (cp. Section 2.4) to a word.

Definition 3.10. *Let A be an alphabet, the* decoration map $\delta : A^* \to (A \times \{0,1,2\})^*$ *is defined as:*

$$\delta(x_1 \dots x_n) = (x_1, q_1)(x_2, q_2) \dots (x_n, q_n).$$

Notation 3.11. *The elements of the image of δ will be called* decorated words*. If a letter $s \in S$ is of color a, i.e. $s \in S_a$, the elements $(s^{\pm 1}, 0)$, $(s^{\pm 1}, 1)$, and $(s^{\pm 1}, 2)$ will also have the color a.*

Example 3.12. *We decorate the word bcabcbcbabcab: $\delta(bcabcbcbabcab) =$*

$$(b,2)(c,1)(a,0)(b,2)(c,0)(b,1)(c,2)(b,1)(a,0)(b,1)(c,2)(a,0)(b,2).$$

In order to apply the decoration to the image of the restriction map $\boldsymbol{\rho_a}$, we define it on tuples. The decoration map $\boldsymbol{\delta}$ on tuples is simply a tuple of decorated entries. Precisely:

$$\boldsymbol{\delta} : (A^*)^m \to ((A \times \{0,1,2\})^*)^m, \quad \boldsymbol{\delta}\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} \delta(v_1) \\ \vdots \\ \delta(v_m) \end{pmatrix}.$$

3.4 Diary map $\boldsymbol{\psi_{c_\lambda}}$

In this section we will consider the *diary map* as it is described in [BDS07]. This is a more elegant formulation than in [DS05], where it was introduced. Moreover, we will adapt this map to our situation.

Let A be a finite alphabet. An element $\alpha \in (A^*)^*$, i.e. $\alpha = \alpha_1 \dots \alpha_k$ with $\alpha_1, \dots, \alpha_k \in A^*$, is called a *sentence* over A.

The diary map takes a sentence over A, i.e. a word over the infinite alphabet A^*, to a word over a finite alphabet Ω with the same length (actually it will be one letter less in our use of the diary map). Thereby we certainly lose information, but if the number of words in the sentence is relatively high, compared to the lengths of words, there remains enough for our purposes.

For technical reasons a *stop sign* is introduced to mark ends of words in a sentence. We use the symbol $\circ$ as the stop sign. Thus we write for a sentence $\alpha = \alpha_1 \dots \alpha_k$:

$$\alpha = \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_k.$$

Unlike the original definition, we do not set the stop sign at the end of the sentence.

The following description explains the diary map figuratively and motivates the naming. There is a girl named Alice, who goes on a journey and keeps a journey diary. On sunny days she has no time for the diary, thus she only writes in her diary on rainy days. More precisely, in the morning of a rainy day Alice writes in her diary: at first she describes the previous day with a letter $s \in S \cup S^{-1} \cup \{\circ\}$, then the day before the previous day and so on until she reported on κ days of her trip, which were not described earlier. Thereby rainy days are denoted by $\circ$. If there are no days left to describe, Alice sets the symbol $\star$ in the diary.

Now we give a formal definition of the diary map ψ and an example illustrating its application on a sentence.

Let $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{k+1}$ be a sentence over A. Let $\kappa > 0$ be an integer, we call it the *diary constant*. We define the *i-th diary entry* ω_i (i-th rainy day) by the following algorithm:

1. Set $w = \alpha_1$.
2. Repeat for $i = 1, \ldots, k$:
   (a) Let be $w = w_m \ldots w_1$ with $w_j \in A$, then define:
   $$\omega_i = \begin{cases} w_1 w_2 \ldots w_\kappa & \text{for } \kappa \le m, \\ w_1 \ldots w_m \star & \text{for } \kappa > m. \end{cases}$$
   (b) Set $w = w_m \ldots w_{\kappa+1} \circ \alpha_{i+1}$.

Thereby ω_i is in the finite set

$$\Omega = (A_\circ)^\kappa \cup \bigcup_{i=0}^{\kappa-1} \left[(A_\circ)^i \star \right] \quad \text{where } A_\circ = A \cup \{\circ\}.$$

Now we can formally define the *diary map* ψ by:

$$\begin{array}{rccl} \psi : & T_{A^*} & \to & T_\Omega \\ & \alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_{k+1} & \mapsto & \omega_1 \omega_2 \ldots \omega_k \end{array}$$

and $\psi(\alpha) = \varepsilon$ if α contains no stop sign.

Obviously we obtain that:

$$\ell(\alpha) - 1 = \ell(\psi(\alpha)).$$

Example 3.13. *Let be* $\alpha = bcabc \circ bc \circ babc \circ a \circ b$, $\kappa = 4$.

1. $w = bcabc$
2.
   - $i = 1$
     (a) $\omega_1 = cbac$
     (b) $w = b \circ bc$
   - $i = 2$
     (a) $\omega_2 = cb \circ b$
     (b) $w = \circ babc$
   - $i = 3$
     (a) $\omega_3 = cbab$
     (b) $w = \circ \circ a$
   - $i = 4$
     (a) $\omega_4 = a \circ \circ \star$
     (b) $w = \circ b$

Thus

$$\psi(\alpha) = (cbac)(cb \circ b)(cbab)(a \circ \circ\star).$$

Remark 3.14. *Let* $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \circ \alpha_{k+1} \cdots \circ \alpha_p$, $\overline{\alpha} = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k \circ \overline{\alpha}_{k+1} \cdots \circ \overline{\alpha}_q$ *be sentences over* A *with coinciding beginnings. Then the diaries* $\psi(\alpha)$ *and* $\psi(\overline{\alpha})$ *also have coinciding beginnings:*

$$\psi(\alpha) = \omega_1 \dots \omega_k \omega_{k+1} \dots \omega_p, \quad \psi(\quad \alpha) = \omega_1 \dots \omega_k \overline{\omega}_{k+1} \dots \overline{\omega}_q.$$

As the idea of a journey diary is to remind one of the journey, the idea of the diary map is to reconstruct the sentence. This is surely only possible in certain settings. For instance, if we reach a $\star$ then we had enough time to write down anything that happened up to this time and thus we can reconstruct it.

Proposition 3.15. *Let* $\alpha = \alpha_1 \circ \cdots \circ \alpha_{k+1}$ *be a sentence over* A *with the diary* $\psi(\alpha) = \omega_1 \dots \omega_k$. *Let* ω_i *contain* $\star$. *Then the subsentence* $\alpha_1 \circ \cdots \circ \alpha_i$ *can be reconstructed out of* $\omega_1 \dots \omega_i$.

Proof: [BDS07], Lemma 4.3. □

Although we omit the proof, the reconstruction procedure is demonstrated by the following example.

Example 3.16. *Let be* $\alpha = bcabc \circ bc \circ babc \circ a \circ b$ *and* $\kappa = 4$ *as in the previous example. The diary entry* ω_4 *contains the sign* $\star$, *thus we can reconstruct* $\alpha_1 \dots \alpha_4$. *The symbol* □ *stands for potentially missing letters.*

- $\omega_1 = cbac \quad \Rightarrow \quad \alpha_1 = \square\, cabc$
- $\omega_2 = cb \circ b \quad \Rightarrow \quad \alpha_1\alpha_2 = \square\, bcabc \circ bc$
- $\omega_3 = cbab \quad \Rightarrow \quad \alpha_1 \dots \alpha_3 = \square\, bcabc \circ bc \square\, babc$
- $\omega_4 = \star \circ \circ a \quad \Rightarrow \quad \alpha_1 \dots \alpha_4 = bcabc \circ bc \circ babc \circ a$

Diary of a word

We apply the diary map ψ to our situation as described in the following. Consider a decorated word $w = w_1 a_1 \dots a_p w_{p+1}$, where $a_1, \dots, a_p$ are exactly the letters of the color a. After every letter a_i we insert a stop sign $\circ$, thus we can apply ψ to the result:

$$\psi_a(w_1 a_1 \dots a_p w_{p+1}) := \psi(w_1 a_1 \circ \dots a_p \circ w_{p+1}).$$

We define the diary map $\boldsymbol{\psi_{c_\lambda}}$ of a tuple of words $(v_1, \dots, v_m)$ as a tuple of diaries of the single words:

$$\boldsymbol{\psi_{c_\lambda}} : \left(((S \cup S^{-1}) \times \{0,1,2\})^*\right)^m \to (\Omega^*)^m, \quad \boldsymbol{\psi_{c_\lambda}} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} \psi_{c_\lambda}(v_1) \\ \vdots \\ \psi_{c_\lambda}(v_m) \end{pmatrix}.$$

For the later proof of the main theorem we need a more detailed reconstruction result than Proposition 3.15. Therefore we modify Lemma 4.4 in [BDS07] and its proof.

Lemma 3.17. *Let $w = w_1 a_1 \dots w_p a_p w_{p+1} \dots w_{p+r} a_{p+r} w_{p+r+1}$ be a word over the alphabet $S \cup S^{-1}$, where $a_1 \dots a_{p+r}$ denote exactly the letters of color a. If*

$$k := (\kappa - 1)r + 1 - \ell(a_{p+1} w_{p+2} a_{p+2} \dots w_{p+r} a_{p+r}) \geq 1,$$

then we can reconstruct the last k letters of the subword w_{p+1} from the a-diary $\psi_a(w) = \omega_1 \dots \omega_{p+r}$ of w. If $\ell(w_{p+1}) < k$, then it can be reconstructed completely.

The following fact results from the reconstruction procedure, which is explained in [BDS07]: A subword $x_1 \dots x_m$ of the word w can be reconstructed from diary entries $\omega_i, \dots, \omega_j$, if these entries contain the letters $x_1, \dots, x_m$. This is used in the proof.

Proof of Lemma 3.17: If one of the diary entries $\omega_{p+1}, \dots, \omega_{p+r}$, say ω_{p+j}, contains $\star$, then the word $w_1 a_1 \dots w_{p+j}$ can be reconstructed by Proposition 3.15.

If one of the diary entries $\omega_{p+1}, \dots, \omega_{p+r}$, say ω_{p+j}, contains a_p, then all the letters of the word w_{p+1} are contained in $\omega_{p+1}, \dots, \omega_{p+j}$ and thus w_{p+1} can be reconstructed.

In the other case diary entries $\omega_{p+1}, \dots, \omega_{p+r}$ contain κr letters of the word $w_{p+1} \dots w_{p+r} a_{p+r}$ and $r-1$ stop signs $\circ$. Thus at least

$$k = \kappa r - (r-1) - \ell(a_{p+1} w_{p+2} a_{p+2} \dots w_{p+2} a_{p+r})$$

of these κr letters are from the word w_{p+1} and by the definition of the diary map, these are the last k letters. □

3.5 Construction of the embedding $\boldsymbol{E}$

Now we combine all the previously defined maps to the embedding $\boldsymbol{E}$ of the group Γ into a product of n trees. For the λ-th component of $\boldsymbol{E}$ we concatenate the maps ν_{c_λ}, $\boldsymbol{\rho_{c_\lambda}}$, $\boldsymbol{\delta}$ and $\boldsymbol{\psi_{c_\lambda}}$:

$$\boldsymbol{E_\lambda} = \boldsymbol{\psi_{c_\lambda}} \circ \boldsymbol{\delta} \circ \boldsymbol{\rho_{c_\lambda}} \circ \nu_{c_\lambda}.$$

This means, we first reduce the normal form of a group element to certain subsets of $S \cup S^{-1}$, then we decorate the result and after all we apply the diary map on the decorated words. More precisely, let γ be in Γ and let a be a color, say c_λ, then:

$$\begin{array}{rcl}
\nu_a(\gamma) & = & w_1 a_1 \dots w_p a_p w_{p+1} =: w \in (S \cup S^{-1})^*, \\
\boldsymbol{\rho_a}(\nu_a(\gamma)) & = & \boldsymbol{\rho_a}(w) \\
 & = & (\rho(w, M \cup S_a \cup S_a^{-1}))_{M \in M_\lambda} =: (v_1, \dots, v_{m_\lambda})^T \in \left((S \cup S^{-1})^*\right)^{m_\lambda}, \\
\boldsymbol{\delta}(\boldsymbol{\rho_a}(\nu_a(\gamma))) & = & \boldsymbol{\delta}((v_1, \dots, v_{m_\lambda})^T) \\
 & = & (\delta(v_1), \dots, \delta(v_{m_\lambda}))^T \in \left(((S \cup S^{-1}) \times \{0,1,2\})^*\right)^{m_\lambda}, \\
\boldsymbol{E_\lambda}(\gamma) & = & \boldsymbol{\psi_a}(\boldsymbol{\delta}(\boldsymbol{\rho_a}(\nu_a(\gamma)))) \\
 & = & \boldsymbol{\psi_a}((\delta(v_1), \dots, \delta(v_{m_\lambda}))^T) \\
 & = & (\psi_a(\delta(v_1)), \dots, \psi_a(\delta(v_{m_\lambda})))^T.
\end{array}$$

Thus

$$\boldsymbol{E_\lambda}(\gamma) = \begin{pmatrix} \psi_a(\delta(v_1)) \\ \vdots \\ \psi_a(\delta(v_{m_\lambda})) \end{pmatrix}$$

consists of the a-diaries of $\delta(v_i)$s. Because the restriction $\boldsymbol{\rho_a}$ preserves the elements of color a, each $\delta(v_i)$ has p letters of color a and hence, $\psi_a(\delta(v_i))$ has p entries.

Thus the element $\psi_a(\delta(v_i)) =: (\omega_{i1}, \dots, \omega_{ip})$ is in Ω^p, where:

$$\Omega = (S_\circ)^\kappa \cup \bigcup_{k=0}^{\kappa-1} \Big[(S_\circ)^k \star \Big] \quad \text{with } S_\circ = (S \cup S^{-1}) \times \{0, 1, 2\} \cup \{\circ\}.$$

The j-th column $\begin{pmatrix} \omega_{1j} \\ \vdots \\ \omega_{m_\lambda j} \end{pmatrix}$ of $\boldsymbol{E_\lambda}(\gamma)$ can also be interpreted as the j-th entry of the a-diary of the vector $\begin{pmatrix} \delta(v_1) \\ \vdots \\ \delta(v_{m_\lambda}) \end{pmatrix}$. Thus $\boldsymbol{E_\lambda}(\gamma)$ can be seen as a "vector valued diary" of γ. Since $\boldsymbol{E_\lambda}(\gamma)$ is in $(\Omega^{m_\lambda})^p$ it can be interpreted as a vertex of the tree $T_{\Omega^{m_\lambda}} =: T_\lambda$ with $|\boldsymbol{E_\lambda}(\gamma)| = p = \ell_a(\gamma)$. Therefore:

$$\boldsymbol{E_\lambda} : \Gamma \to T_\lambda.$$

Note that Ω^{m_λ} is a finite alphabet, thus T_λ can be quasi-isometrically embedded into a binary tree (Remark 2.3).

Definition 3.18. *Let $C(\Gamma, S)$ be a Cayley graph of an n-colored right-angled Artin or Coxeter group Γ with a system of generators S. Let $\kappa \in \mathbb{N}$ be a constant. The map $\boldsymbol{E} : C(\Gamma, S) \to \Pi_{i=1}^n T_i$ is defined as:*

$$\boldsymbol{E}(\gamma) = (\boldsymbol{E_1}(\gamma), \dots, \boldsymbol{E_n}(\gamma)),$$

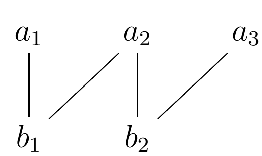
where $\boldsymbol{E_i} = \boldsymbol{\psi_{c_i}} \circ \boldsymbol{\delta} \circ \boldsymbol{\rho_{c_i}} \circ \nu_{c_i}$.

Example

As an example of the map $\boldsymbol{E}$ consider the 2-colored right-angled Coxeter group

$$\Gamma = \big\langle a_1, a_2, a_3, b_1, b_2 \mid a_i^2 = b_i^2 = 1,\, [a_1 b_1] = [a_2 b_1] = [a_2 b_2] = [a_3 b_2] = 1 \big\rangle$$

with the defining graph:



The colors are $a := c_1$ and $b := c_2$ with $a < b$, a_i are of the color a and b_i of the color b:

$$S_a = \{a_1, a_2, a_3\}, \quad S_b = \{b_1, b_2\}.$$

We choose the diary constant κ as 3.

Consider the element

$$\gamma = \pi(a_1 b_1 a_2 a_1 b_2 a_3 b_1 a_1).$$

First we construct $\boldsymbol{E}_1(\gamma) = \boldsymbol{\psi_a} \circ \boldsymbol{\delta} \circ \boldsymbol{\rho_a} \circ \nu_a(\gamma)$.

The canonical a-representation $\nu_a(\gamma)$ of the element γ is the following:

$$\nu_a(\gamma) = a_1 a_2 a_1 b_1 a_3 b_2 a_1 b_1.$$

The letters of color a are here as far left as possible.

Now we apply the reduction map $\boldsymbol{\rho_a}$ to $\nu_a(\gamma)$. We reduce the word $\nu_a(\gamma)$ to subwords in letters S_a of color a and subsets of S_b with one or two elements, which are: $\{b_1\}$, $\{b_2\}$ and $\{b_1, b_2\}$.

$$\begin{aligned}
\boldsymbol{\rho_a} \circ \nu_a(\gamma) &= \begin{pmatrix} \rho(\nu_a(\gamma), \{a_1, a_2, a_3\} \cup \{b_1\}) \\ \rho(\nu_a(\gamma), \{a_1, a_2, a_3\} \cup \{b_2\}) \\ \rho(\nu_a(\gamma), \{a_1, a_2, a_3\} \cup \{b_1, b_2\}) \end{pmatrix} \\
&= \begin{pmatrix} a_1\, a_2\, a_1\, b_1\, a_3\, a_1\, b_1 \\ a_1\, a_2\, a_1\, a_3\, b_2\, a_1 \\ a_1\, a_2\, a_1\, b_1\, a_3\, b_2\, a_1\, b_1 \end{pmatrix}
\end{aligned}$$

Then we decorate each entry of $\boldsymbol{\rho_a} \circ \nu_a(\gamma)$ with the square-free sequence:

$$(q_i)_{i \in \mathbb{N}} = 21020121012\ldots$$

Thus:

$$\begin{aligned}
\boldsymbol{\delta} \circ \boldsymbol{\rho_a} \circ \nu_a(\gamma) &= \begin{pmatrix} \delta(a_1\, a_2\, a_1\, b_1\, a_3\, a_1\, b_1) \\ \delta(a_1\, a_2\, a_1\, a_3\, b_2\, a_1\,) \\ \delta(a_1\, a_2\, a_1\, b_1\, a_3\, b_2\, a_1\, b_1) \end{pmatrix} \\
&= \begin{pmatrix} (a_1, 2)\, (a_2, 1)\, (a_1, 0)\, (b_1, 2)\, (a_3, 0)\, (a_1, 1)\, (b_1, 2) \\ (a_1, 2)\, (a_2, 1)\, (a_1, 0)\, (a_3, 2)\, (b_2, 0)\, (a_1, 1) \\ (a_1, 2)\, (a_2, 1)\, (a_1, 0)\, (b_1, 2)\, (a_3, 0)\, (b_2, 1)\, (a_1, 2)\, (b_1, 1) \end{pmatrix}.
\end{aligned}$$

The last operation is the diary map $\boldsymbol{\psi_a}$. We apply the map ψ_a to each entry of $\boldsymbol{\delta} \circ \boldsymbol{\rho_a} \circ \nu_a(\gamma)$, which means, we first put the stop signs $\circ$ behind the letters of color a and then apply the diary map ψ with the diary constant $\kappa = 3$:

$$\begin{aligned}
& \boldsymbol{E_1}(\gamma) \\
= \; & \boldsymbol{\psi_a} \circ \boldsymbol{\delta} \circ \boldsymbol{\rho_a} \circ \nu_a(\gamma) \\
= \; & \begin{pmatrix} \psi_a((a_1,2)\,(a_2,1)\,(a_1,0)\,(b_1,2)\,(a_3,0)\,(a_1,1)\,(b_1,2)) \\ \psi_a((a_1,2)\,(a_2,1)\,(a_1,0)\,(a_3,2)\,(b_2,0)\,(a_1,1)) \\ \psi_a((a_1,2)\,(a_2,1)\,(a_1,0)\,(b_1,2)\,(a_3,0)\,(b_2,1)\,(a_1,2)\,(b_1,1)) \end{pmatrix} \\
= \; & \begin{pmatrix} \psi((a_1,2)\circ(a_2,1)\circ(a_1,0)\circ(b_1,2)\,(a_3,0)\circ(a_1,1)\circ(b_1,2)) \\ \psi((a_1,2)\circ(a_2,1)\circ(a_1,0)\circ(a_3,2)\circ(b_2,0)\,(a_1,1)\circ) \\ \psi((a_1,2)\circ(a_2,1)\circ(a_1,0)\circ(b_1,2)\,(a_3,0)\circ(b_2,1)\,(a_1,2)\circ(b_1,1)) \end{pmatrix} \\
= \; & \begin{pmatrix} ((a_1,2)\star) & ((a_2,1)\circ\star) & ((a_1,0)\circ\star) & ((a_3,0)\,(b_1,2)\circ) & ((a_1,1)\circ\star) \\ ((a_1,2)\star) & ((a_2,1)\circ\star) & ((a_1,0)\circ\star) & ((a_3,2)\circ\star) & ((a_1,1)\,(b_2,0)\circ) \\ ((a_1,2)\star) & ((a_2,1)\circ\star) & ((a_1,0)\circ\star) & ((a_3,0)\,(b_1,2)\circ) & ((a_1,2)\,(b_2,1)\circ) \end{pmatrix}.
\end{aligned}$$

Each column of $\boldsymbol{E_1}(\gamma)$ is in $\Omega^{m_1} = \Omega^3$ with

$$\Omega = (S_\circ)^3 \cup \bigcup_{k=0}^{2} \left[(S_\circ)^k \star \right] \quad \text{with } S_\circ = S \times \{0,1,2\} \cup \{\circ\}.$$

Hence $\boldsymbol{E_1}(\gamma)$ is in $(\Omega^{m_1})^{\ell_a(\gamma)} = (\Omega^3)^5$ and, therefore, $\boldsymbol{E_1}(\gamma) \in T_1 = T_{\Omega^3}$.

For constructing $\boldsymbol{E_2}(\gamma)$ we repeat the same procedure with color b.

The canonical b-representation $\nu_b(\gamma)$ of the element γ is the following:

$$\nu_b(\gamma) = b_1 a_1 a_2 a_1 b_2 a_3 b_1 a_1.$$

Now the letters of color b are as far left as possible.

The reduction map $\boldsymbol{\rho_b}$ reduces the word $\nu_b(\gamma)$ to subwords in letters S_b of color b and subsets of S_a with one or two elements, which are: $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_1, a_2\}$, $\{a_1, a_3\}$ and $\{a_2, a_3\}$.

$$\begin{aligned}
\boldsymbol{\rho_b} \circ \nu_b(\gamma) &= \begin{pmatrix} \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_1\}) \\ \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_2\}) \\ \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_3\}) \\ \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_1,a_2\}) \\ \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_1,a_3\}) \\ \rho(\nu_b(\gamma), \{b_1,b_2\} \cup \{a_2,a_3\}) \end{pmatrix} \\
&= \begin{pmatrix} b_1\,a_1\,a_1\,b_2\,b_1\,a_1 \\ b_1\,a_2\,b_2\,b_1 \\ b_1\,b_2\,a_3\,b_1 \\ b_1\,a_1\,a_2\,a_1\,b_2\,b_1\,a_1 \\ b_1\,a_1\,a_1\,b_2\,a_3\,b_1\,a_1 \\ b_1\,a_2\,b_2\,a_3\,b_1 \end{pmatrix}
\end{aligned}$$

Here we remark that $\boldsymbol{\rho_b} \circ \nu_b(\gamma)$ actually consists of words, sequences of letters without any relation. Thus the subword $a_1\, a_1$ in the first entry can not be removed by reason that $(a_1)^2 = 1$ in a Coxeter group.

Then we decorate each entry of $\boldsymbol{\rho_b} \circ \nu_b(\gamma)$ with the square-free sequence $(q_i)_{i\in\mathbb{N}} = 21020121012\ldots$

$$\begin{aligned}
\boldsymbol{\delta} \circ \boldsymbol{\rho_b} \circ \nu_b(\gamma) &= \begin{pmatrix} \delta(b_1\, a_1\, a_1\, b_2\, b_1\, a_1) \\ \delta(b_1\, a_2\, b_2\, b_1) \\ \delta(b_1\, b_2\, a_3\, b_1) \\ \delta(b_1\, a_1\, a_2\, a_1\, b_2\, b_1\, a_1) \\ \delta(b_1\, a_1\, a_1\, b_2\, a_3\, b_1\, a_1) \\ \delta(b_1\, a_2\, b_2\, a_3\, b_1) \end{pmatrix} \\
&= \begin{pmatrix} (b_1,2)\,(a_1,1)\,(a_1,0)\,(b_2,2)\,(b_1,0)\,(a_1,1) \\ (b_1,2)\,(a_2,1)\,(b_2,0)\,(b_1,2) \\ (b_1,2)\,(b_2,1)\,(a_3,0)\,(b_1,2) \\ (b_1,2)\,(a_1,1)\,(a_2,0)\,(a_1,2)\,(b_2,0)\,(b_1,1)\,(a_1,2) \\ (b_1,2)\,(a_1,1)\,(a_1,0)\,(b_2,2)\,(a_3,0)\,(b_1,1)\,(a_1,2) \\ (b_1,2)\,(a_2,1)\,(b_2,0)\,(a_3,2)\,(b_1,0) \end{pmatrix}
\end{aligned}$$

We apply the map ψ_b to each entry of $\boldsymbol{\delta} \circ \boldsymbol{\rho_b} \circ \nu_b(\gamma)$, we first put the stop signs $\circ$ behind the letters of color b and then apply the diary map ψ with the diary constant $\kappa = 3$:

$$\begin{aligned}
\boldsymbol{E_2}(\gamma) &= \boldsymbol{\psi_b} \circ \boldsymbol{\delta} \circ \boldsymbol{\rho_b} \circ \nu_b(\gamma) \\
&= \begin{pmatrix} \psi_b((b_1,2)\,(a_1,1)\,(a_1,0)\,(b_2,2)\,(b_1,0)\,(a_1,1)) \\ \psi_b((b_1,2)\,(a_2,1)\,(b_2,0)\,(b_1,2)) \\ \psi_b((b_1,2)\,(b_2,1)\,(a_3,0)\,(b_1,2)) \\ \psi_b((b_1,2)\,(a_1,1)\,(a_2,0)\,(a_1,2)\,(b_2,0)\,(b_1,1)\,(a_1,2)) \\ \psi_b((b_1,2)\,(a_1,1)\,(a_1,0)\,(b_2,2)\,(a_3,0)\,(b_1,1)\,(a_1,2)) \\ \psi_b((b_1,2)\,(a_2,1)\,(b_2,0)\,(a_3,2)\,(b_1,0)) \end{pmatrix} \\
&= \begin{pmatrix} \psi((b_1,2)\circ(a_1,1)\,(a_1,0)\,(b_2,2)\circ(b_1,0)\circ(a_1,1)) \\ \psi((b_1,2)\circ(a_2,1)\,(b_2,0)\circ\,(b_1,2)\circ) \\ \psi((b_1,2)\circ(b_2,1)\circ(a_3,0)\,(b_1,2)\circ) \\ \psi((b_1,2)\circ(a_1,1)\,(a_2,0)\,(a_1,2)\,(b_2,0)\circ(b_1,1)\circ(a_1,2)) \\ \psi((b_1,2)\circ(a_1,1)\,(a_1,0)\,(b_2,2)\circ(a_3,0)\,(b_1,1)\circ(a_1,2)) \\ \psi((b_1,2)\circ(a_2,1)\,(b_2,0)\circ(a_3,2)\,(b_1,0)\circ) \end{pmatrix} \\
&= \begin{pmatrix} ((b_1,2)\star) & ((b_2,2)\,(a_1,0)\,(a_1,1)) & ((b_1,0)\circ\circ) \\ ((b_1,2)\star) & ((b_2,0)\,(a_2,1)\circ) & ((b_1,2)\circ\star) \\ ((b_1,2)\star) & ((b_2,1)\circ\star) & ((b_1,2)\,(a_3,0)\circ) \\ ((b_1,2)\star) & ((b_2,0)\,(a_1,2)\,(a_2,0)) & ((b_1,1)\circ(a_1,1)) \\ ((b_1,2)\star) & ((b_2,2)\,(a_1,0)\,(a_1,1)) & ((b_1,1)\,(a_3,0)\circ) \\ ((b_1,2)\star) & ((b_2,0)\,(a_2,1)\circ) & ((b_1,0)\,(a_3,2)\circ) \end{pmatrix}.
\end{aligned}$$

Each column of $\boldsymbol{E_2}(\gamma)$ is in $\Omega^{m_2} = \Omega^6$ with

$$\Omega = (S_\circ)^3 \cup \bigcup_{k=0}^{2} \Big[(S_\circ)^k \star \Big] \quad \text{with } S_\circ = S \times \{0,1,2\} \cup \{\circ\}.$$

Hence $\boldsymbol{E_2}(\gamma)$ is in $(\Omega^{m_2})^{\ell_b(\gamma)} = (\Omega^6)^3$ and, therefore, $\boldsymbol{E_2}(\gamma) \in T_2 = T_{\Omega^6}$.

Altogether:

$$\boldsymbol{E}(\gamma) = (\boldsymbol{E}_1(\gamma), \boldsymbol{E}_2(\gamma)) \in T_1 \times T_2.$$

3.6 Necessity of the decoration map $\boldsymbol{\delta}$

In this section, we want to explain why the decoration map is essential for the embedding, precisely we will show that the undecorated diary map is not quasi-isometric. We disregard the restriction part $\boldsymbol{\rho_{c_\lambda}}$ of the embedding because it does not concern the problem, which is the following: The diary map reconstructs the words from the end but it can happen that the words are essentially different but coincide in the reconstructible (back) part. This problem occurs e.g. in case of periodic words as in the following example from [DS05].

We first describe the idea and then give a concrete example. Consider the words A and $B \in S^*$ in the generators S of a right-angled Coxeter group Γ with at least two colors a and b of generators, which fulfill the following:

a. The word A is formed by a-letters.

b. The word B is formed by b-letters and the first and the last letter of B are different and do not commute with the first letter of A.

Moreover, $\ell(B) \ll \ell(A)$, which means that the length of B is essentially smaller than the length of A. We take a $k \in \mathbb{N}$ such that $\ell(A) \ll k$ and the diary constant $\kappa \ll k$.

Consider the elements

$$\gamma = \pi(B^k A) \quad \text{and} \quad \overline{\gamma} = \pi(B^{k+1} A).$$

The a- and the b-representations of these elements coincide:

$$\nu_a(\gamma) = \nu_b(\gamma) = B^k A$$
$$\nu_a(\overline{\gamma}) = \nu_b(\overline{\gamma}) = B^{k+1} A.$$

The a-diaries of γ and $\overline{\gamma}$ are the same because k is chosen large enough such that B^k can not be reconstructed completely and B^k and B^{k+1} have the same reconstructible endings:

$$|\psi_a(\nu_a(\gamma))\psi_a(\nu_a(\overline{\gamma}))| = 0.$$

The b-diaries have the first $k\ell(B)$ entries in common. Besides the b-diary of $\overline{\gamma}$ has $\ell(B)$ additional entries, thus the b-diaries have distance $\ell(B)$:

$$|\psi_b(\nu_b(\gamma))\psi_b(\nu_b(\overline{\gamma}))| = \ell(B).$$

Together, it follows:

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 = |\psi_a(\nu_a(\gamma))\psi_a(\nu_a(\overline{\gamma}))| + |\psi_b(\nu_b(\gamma))\psi_b(\nu_b(\overline{\gamma}))| = \ell(B).$$

Contrarily the distance of the elements γ and $\overline{\gamma}$ is essentially larger. Since

$$\gamma^{-1}\overline{\gamma} = \pi(A^{-1}(B^k)^{-1}B^{k+1}A) = \pi(A^{-1}BA)$$

and the word $A^{-1}BA$ is reduced, we have:

$$d(\gamma, \overline{\gamma}) = \ell(A^{-1}BA) = 2\ell(A) + \ell(B).$$

Because $\ell(B) \ll \ell(A)$ it follows:

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 \ll d(\gamma, \overline{\gamma}).$$

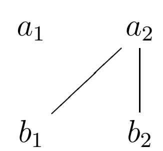
Since $\ell(A)$ can be chosen arbitrary large, the distance $d(\gamma, \overline{\gamma})$ of the group elements is arbitrary longer then the distance $|\psi(\gamma)\psi(\overline{\gamma})|_1$ of their images. Therefore the undecorated diary map ψ is not quasi-isometric.

The decoration map makes sure that the k-th and the $k+1$-th B in B^{k+1} differ in the decoration because it is square-free and that therefore the a-diaries differ. Thus the decoration counteract the periodicity.

A concrete example is a right angled 2-colored Coxeter group

$$\Gamma = \left\langle a_1, a_2, b_1, b_2 \mid a_i^2 = b_i^2 = 1,\ [a_2 b_1] = [a_2 b_2] = 1 \right\rangle$$

with the defining graph:



The words

$$A = a_1 a_2 a_1 \dots a_1 a_2,$$
$$B = b_1 b_2 b_1 \dots b_1 b_2$$

are chosen with appropriate length.

3.7 Necessity of the restriction map ρ_{c_λ}

The restriction map reduces words to subwords in a smaller number of generators to identify the concrete differences. Consider the decorated diary map without reduction. It can happen that the reconstructible coinciding parts of two different words commute with the non-reconstructible differing parts, i.e. it depends on the order of commuting generators if we can reconstruct the differing parts. The reduction map "cancels" the coinciding generators and concentrates on the differing.

We first explain the idea of the counter-example, where the differing parts are not reconstructible, and afterwards we look at a concrete group with the required properties.

Thus consider a finitely generated 3-colored right-angled Coxeter group with the generator set S. We denote the colors of S by a, b and c. Consider two elements

$$\gamma = \pi(BCa_1A) \quad \text{and} \quad \overline{\gamma} = \pi(B\overline{B}Ca_1A)$$

where $a_1, A, B, \overline{B}, C$ are chosen as follows:

a. The word A is formed by a-letters, a_1 is an a-letter.

b. The words B and $\overline{B} \neq \varepsilon$ have the color b and the last κ letters of B and $B\overline{B}$ coincide even with the square-free decoration. The last letter of B (and $\overline{B}$) does not commute with a_1.

c. The word C has the color c and it commutes with $a_1, B, \overline{B}$. The last letter of C does not commute with the first letter of A.

Consider additionally that $\ell(\overline{B}) \ll \ell(A)$ and $\ell(A) \ll \ell(C)$. We will show that

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 \ll d(\gamma^{-1}\overline{\gamma}), \tag{3.1}$$

which is a contradiction to the quasi-isometry property.

We first investigate the diary maps by the colors. The canonical a-representations of the words are the following:

$$\nu_a(\gamma) = Ba_1CA, \quad \nu_a(\overline{\gamma}) = B\overline{B}a_1CA.$$

The diary entries of a_1 in γ and $\overline{\gamma}$ are the last κ decorated letters of B and $\overline{B}$, respectively, so they are the same by choice of $B, \overline{B}$. The diary entries of A are also the same, because they only reconstruct parts of C and A but not of B or $\overline{B}$ by reason of the length of C. Hence $|\psi_a(\nu_a(\gamma))\psi_a(\nu_a(\overline{\gamma}))| = 0$.

The canonical b-representations of the words are the following:

$$\nu_b(\gamma) = BD, \quad \nu_b(\overline{\gamma}) = B\overline{B}D, \qquad \text{where } D \text{ is the word } Ca_1A \text{ in a certain order.}$$

The first $\ell(B)$ entries are the same. Now $\psi_b(\nu_b(\overline{\gamma}))$ has $\ell(\overline{B})$ additional entries, so $|\psi_b(\nu_b(\gamma))\psi_b(\nu_b(\overline{\gamma}))| = \ell(\overline{B})$.

The canonical c-representations of the words are:

$$\nu_c(\gamma) = CBa_1A, \quad \nu_c(\overline{\gamma}) = CB\overline{B}a_1A.$$

The diary entries of C are the same and therefore $|\psi_c(\nu_c(\gamma))\psi_c(\nu_c(\overline{\gamma}))| = 0$.

Altogether we have

$$\begin{aligned} & |\psi(\gamma)\psi(\overline{\gamma})|_1 \\ = \; & |\psi_a(\nu_a(\gamma))\psi_a(\nu_a(\overline{\gamma}))| + |\psi_b(\nu_b(\gamma))\psi_b(\nu_b(\overline{\gamma}))| + |\psi_c(\nu_c(\gamma))\psi_c(\nu_c(\overline{\gamma}))| \\ = \; & \ell(\overline{B}). \end{aligned}$$

Now consider the distance between γ and $\overline{\gamma}$. Because

$$\gamma^{-1}\overline{\gamma} = \pi(A^{-1}a_1C^{-1}B^{-1}BC\overline{B}a_1A) = \pi(A^{-1}a_1\overline{B}a_1A)$$

and the word $A^{-1}a_1\overline{B}a_1A$ is reduced, it holds:

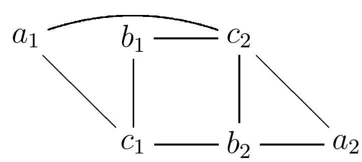
$$d(\gamma, \overline{\gamma}) = \ell(A^{-1}a_1\overline{B}a_1A) = 2\ell(A) + 2 + \ell(\overline{B}).$$

Consequently we can show (3.1):

$$|\psi(\gamma)\psi(\overline{\gamma})|_1 = \ell(\overline{B}) \ll \ell(A) \leq 2\ell(A) + 2 + \ell(\overline{B}) = d(\gamma^{-1}\overline{\gamma}).$$

More concretely, we can look at a minimal example of such group. Consider the group $\Gamma = \big\langle a_1, a_2, b_1, b_2, c_1, c_2 \mid a_1^2 = a_2^2 = b_1^2 = b_2^2 = c_1^2 = c_2^2 = 1, [a_1c_1] = [a_1c_2] = [a_2b_2] = [a_2c_2] = [b_1c_1] = [b_1c_2] = [b_2c_1] = [b_2c_2] = 1\big\rangle$.

The defining graph of Γ is:



And the words we need are:

$$\begin{aligned} A &= a_2a_1a_2 \ldots a_2a_1, \\ B &= b_2b_1b_2 \ldots b_2b_1, \\ \overline{B} &= b_2b_1b_2 \ldots b_2b_1, \\ C &= c_2c_1c_2 \ldots c_2c_1. \end{aligned}$$

Choose the length of $B, \overline{B}$ such that the decoration of the last κ letters of B is equal to the decoration of the last κ letters of $\overline{B}$ (this is possible for combinatorial reasons). The words A and C can be chosen arbitrary long.

In the next chapter we will prove that the map $\boldsymbol{E}$ is a quasi-isometric embedding as desired.

Chapter 4

Proof of Theorem 1.1

In this chapter we prove that the map $\boldsymbol{E}$ is a quasi-isometric embedding. More precisely:

Proposition 4.1. *Let Γ be a finitely generated n-colored right-angled Artin or Coxeter group with a generator system S, let $\boldsymbol{E}$ be the map defined in 3.18 with $\kappa > 3n$. Then $\boldsymbol{E}$ is a quasi-isometric embedding of the Cayley graph $C(\Gamma, S)$ of Γ into the product $\prod_{i=1}^{n} T_i$ of trees T_i with uniformly bounded valence.*

Since the tree T_i can be quasi-isometrically embedded into a binary tree, see Remark 2.3, Theorem 1.1 results immediately from this statement.

Recall that the map $\boldsymbol{E}$ is a quasi-isometric embedding if there are $\alpha \geq 1$, $\beta \geq 0$ such that for all $\gamma, \overline{\gamma} \in \Gamma$:

$$\frac{1}{\alpha} d(\gamma, \overline{\gamma}) - \beta \leq |\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1 \leq \alpha d(\gamma, \overline{\gamma}) + \beta. \tag{4.1}$$

We first show the second inequality and then the first.

Lemma 4.2. *The map $\boldsymbol{E}$ is 1-Lipschitz, in particular the second part of the inequality (4.1) is true.*

This is basically the proof of Lemma 3.6 in [DS05] adapted to the differing embedding $\boldsymbol{E}$.

Proof: We first consider two elements $\gamma, \overline{\gamma} \in \Gamma$ with distance $d(\gamma, \overline{\gamma}) = 1$. Assume without loss of generality $\ell(\overline{\gamma}) > \ell(\gamma)$. This means that there is a reduced word w with $\pi(w) = \gamma$ and there is a generator $s \in S$ such that $\pi(ws) = \overline{\gamma}$. Assume that s is of the color c_i. Then diaries with respect to other colors coincide:

$$\boldsymbol{E_j}(\gamma) = \boldsymbol{E_j}(\overline{\gamma}) \quad \text{for} \quad j \neq i.$$

The c_i-diary $\boldsymbol{E_i}(\overline{\gamma})$ of $\overline{\gamma}$ has one entry more than the c_i-diary $\boldsymbol{E_i}(\gamma)$ of γ and the other entries are the same:

$$\boldsymbol{E_i}(\gamma) = \omega_{i1}, \ldots, \omega_{ip}, \quad \boldsymbol{E_i}(\overline{\gamma}) = \omega_{i1}, \ldots, \omega_{ip}, \omega_{ip+1}.$$

It follows $|\boldsymbol{E_i}(\gamma)\boldsymbol{E_i}(\overline{\gamma})| = 1$ and $|\boldsymbol{E_j}(\gamma)\boldsymbol{E_j}(\overline{\gamma})| = 0$ for $j \neq i$. Hence:

$$|\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1 = \sum_{k=1}^{n} |\boldsymbol{E_k}(\gamma)\boldsymbol{E_k}(\overline{\gamma})| = 1.$$

Now we consider the general situation $d(\gamma, \overline{\gamma}) = k$. There is a geodesic segment in the Cayley graph of Γ given by a sequence $\gamma = \gamma_0, \ldots, \gamma_k = \overline{\gamma}$ with $d(\gamma_i, \gamma_{i-1}) = 1$ for $i \in 1, \ldots, k$. It follows from the equation above and the triangle inequality:

$$|\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1 \leq \sum_{i=1}^{k} |\boldsymbol{E}(\gamma_i)\boldsymbol{E}(\gamma_{i-1})|_1 = k = d(\gamma, \overline{\gamma}).$$

□

Proof of Proposition 4.1: We follow the scheme used in [DS05] to prove Theorem 3.11. However, we use different arguments for significant aspects.

We show the lower estimate of the inequality (4.1).

Let be $\gamma, \overline{\gamma} \in \Gamma$ and $c := d(\gamma, \overline{\gamma})$.

Suppose $c < 30n$, then:

$$\frac{c}{3n} - 10 < 0 \leq |\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1.$$

The proposition is proven, if $c \geq 30n$ implies

$$\frac{c}{3n} \leq |\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1.$$

Thus, assume $c \geq 30n$.

Notations:

Let $a := c_\lambda$ be a most common color of letters in $\gamma^{-1}\overline{\gamma}$, i.e. $\ell_a(\gamma^{-1}\overline{\gamma}) \geq \ell_b(\gamma^{-1}\overline{\gamma})$ for all colors b. Since there are n colors:

$$\ell_a(\gamma^{-1}\overline{\gamma}) \geq \frac{c}{n}. \tag{4.2}$$

By Lemma 2.7, elements 1, γ, and $\overline{\gamma}$ span a tripod with a midpoint δ. We introduce the following notations for the a-representations of these elements:

$$\begin{aligned}
\nu_a(\delta) &= u = u_1 a_1 \ldots a_p u_{p+1}, \\
\nu_a(\delta^{-1}\gamma) &= v = v_{p+1} a_{p+1} \ldots a_{p+m} v_{p+m+1}, \\
\nu_a(\delta^{-1}\overline{\gamma}) &= \overline{v} = \overline{v}_{p+1} \overline{a}_{p+1} \ldots \overline{a}_{p+\overline{m}} \overline{v}_{p+\overline{m}+1}, \\
\nu_a(\gamma) &= w = w_1 a_1 \ldots a_{p+m} w_{p+m+1}, \\
\nu_a(\overline{\gamma}) &= \overline{w} = \overline{w}_1 \overline{a}_1 \ldots \overline{a}_{p+\overline{m}} \overline{w}_{p+\overline{m}+1}.
\end{aligned}$$

Furthermore, by Lemma 2.7 the word $v^{-1}\overline{v}$ is reduced and $\pi(v^{-1}\overline{v}) = \gamma^{-1}\overline{\gamma}$.

By Corollary 3.5 it holds for $1 \leq i \leq p$ that $w_i a_i = u_i a_i = \overline{w}_i \overline{a}_i$. Thus w and $\overline{w}$ coincide up to a_p:

$$w = u_1 a_1 \dots u_p a_p w_{p+1} a_{p+1} \dots a_{p+m} w_{p+m+1}, \tag{4.3}$$
$$\overline{w} = u_1 a_1 \dots u_p a_p \overline{w}_{p+1} \overline{a}_{p+1} \dots \overline{a}_{p+\overline{m}} \overline{w}_{p+\overline{m}+1}. \tag{4.4}$$

With this notation the inequality (4.2) can be written as:

$$m + \overline{m} \geq \frac{c}{n}. \tag{4.5}$$

Now we still have to prove that for $c \geq 30n$ the inequality $|\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1 \geq \frac{c}{3n}$ holds.

Assume the opposite:

$$|\boldsymbol{E}(\gamma)\boldsymbol{E}(\overline{\gamma})|_1 < \frac{c}{3n}. \tag{4.6}$$

Thus in particular the distance $|\boldsymbol{E}_{\boldsymbol{\lambda}}(\gamma)\boldsymbol{E}_{\boldsymbol{\lambda}}(\overline{\gamma})|$ between $\boldsymbol{E}_{\boldsymbol{\lambda}}(\gamma) = \omega_1 \dots \omega_{p+m}$ and $\boldsymbol{E}_{\boldsymbol{\lambda}}(\overline{\gamma}) = \overline{\omega}_1 \dots \overline{\omega}_{p+\overline{m}}$ is smaller than $\frac{c}{3n}$. Therefore with (4.5) there exists $r \geq \frac{c}{3n}$ such that:

$$\omega_1 \dots \omega_{p+r} = \overline{\omega}_1 \dots \overline{\omega}_{p+r}. \tag{4.7}$$

Hence the a-diary entries ω_i and $\overline{\omega}_i$ coincide for all $1 \leq i \leq p + r$. In particular $a_i = \overline{a}_i$. Note that $\omega_i, \overline{\omega}_i \in \Omega^{m_\lambda}$.

Case 1: The words v_{p+1} and $\overline{v}_{p+1}$ are empty: $v_{p+1} = \varepsilon = \overline{v}_{p+1}$.

As seen above $a_{p+1} = \overline{a}_{p+1}$. Thus $a_{p+1}^{-1}\overline{a}_{p+1} = 1 \in \Gamma$ and therefore the word $v^{-1}\overline{v} = \dots a_{p+1}^{-1}\overline{a}_{p+1} \dots$ is not reduced, which is a contradiction to the choice of v and $\overline{v}$.

Case 2: The words v_{p+1} and $\overline{v}_{p+1}$ are not both empty. As before, $v^{-1}\overline{v}$ is reduced. Then $v_{p+1}^{-1}\overline{v}_{p+1}$ is also reduced. Therefore $v_{p+1} \neq \overline{v}_{p+1}$. Assume without loss of generality $\ell = \ell(v_{p+1}) \geq \ell(\overline{v}_{p+1}) = \overline{\ell}$.

Take the last differing letters x_i and $\overline{x}_i$ of

$$v_{p+1} = x_\ell \dots x_1 \quad \text{and} \quad \overline{v}_{p+1} = \overline{x}_{\overline{\ell}} \dots \overline{x}_1.$$

More precisely: $x_i \neq \overline{x}_i$ and $x_j = \overline{x}_j$ for all $j < i$. If $x_j = \overline{x}_j$ for all $j \leq \overline{\ell}$, then let $i = \overline{\ell} + 1$ and $\overline{x}_i = \varepsilon$.

Now we distinguish two cases with regard to the number of letters

$$x_i =: s \in S \cup S^{-1} \text{ and } \overline{x}_i =: t \in S \cup S^{-1} \cup \{\varepsilon\}$$

in v_{p+1} and $\overline{v}_{p+1}$, which we denote by ℓ_s and ℓ_t respectively.

Case 2.1: There are more letters of one kind in one of the words:

$$\begin{array}{rll} (i) & \ell_s(v_{p+1}) > \ell_s(\overline{v}_{p+1}) & \text{or} \\ (ii) & \ell_s(v_{p+1}) < \ell_s(\overline{v}_{p+1}) & \text{or} \\ (iii) & \ell_t(v_{p+1}) > \ell_t(\overline{v}_{p+1}) & \text{or} \\ (iv) & \ell_t(v_{p+1}) < \ell_t(\overline{v}_{p+1}). & \end{array}$$

Note that if $\overline{x}_i = \varepsilon$ then (i) applies.

We will only prove (i), the other cases are similar.

Let be $\eta := \ell_s(v_{p+1})$ and $\overline{\eta} := \ell_s(\overline{v}_{p+1})$. We denote by $\{s,a\}$ the set of all generators of color a and the generator s:

$$\{s,a\} := \{s\} \cup S_a \cup S_a^{-1}.$$

In the word w_{p+1}, there are no letters of color a. Thus:

$$[w_{p+1}]^{\{s,a\}} = [w_{p+1}]^{\{s\}}.$$

Since $x_i = s$, Lemma 3.9 (2) provides:

$$\begin{aligned} [w_{p+1}]^{\{s\}} &= [u_{p+1}]^{\{s\}}[v_{p+1}]^{\{s\}} \\ &= [u_{p+1}]^{\{s\}} s^{\ell_s(v_{p+1})} \\ &= [u_{p+1}]^{\{s\}} s^{\eta}. \end{aligned}$$

Then:

$$[w_{p+1}]^{\{s,a\}} = [u_{p+1}]^{\{s\}} s^{\eta}. \tag{4.8}$$

The reduction of the word $w_1a_1 \ldots a_pw_{p+1}$ to the set $\{s,a\}$ has the form:

$$\begin{aligned} [w_1a_1 \ldots a_pw_{p+1}]^{\{s,a\}} &\overset{(4.3)}{=} [u_1a_1 \ldots u_pa_pw_{p+1}]^{\{s,a\}} \\ &\overset{\text{Rem. } 3.7}{=} [u_1a_1 \ldots u_pa_p]^{\{s,a\}}[w_{p+1}]^{\{s,a\}} \\ &\overset{(4.8)}{=} [u_1a_1 \ldots u_pa_p]^{\{s,a\}}[u_{p+1}]^{\{s\}} s^{\eta} \\ &\overset{\text{Rem. } 3.7}{=} [u]^{\{s,a\}} s^{\eta}. \end{aligned}$$

Because $x_i = s$ and v is in a-representation, there is a letter out of $x_{i-1}, \ldots, x_1, a_{p+1}$ which does not commute with s. Since

$$\overline{v}_{p+1} = \overline{x}_{\overline{\ell}} \ldots \overline{x}_i x_{i-1} \ldots x_1,$$

the assumptions of Lemma 3.9 (1) apply to the word $\overline{w}$, hence:

$$[\overline{w}_{p+1}]^{\{s\}} = [u_{p+1}]^{\{s\}}[\overline{v}_{p+1}]^{\{s\}}.$$

Thus we obtain in the same way as before for w an analogous result for $\overline{w}$:

$$[\overline{w}_1\overline{a}_1 \dots \overline{w}_{p+1}]^{\{s,a\}} = [u]^{\{s,a\}}s^{\overline{\eta}}.$$

Denote $\ell([u]^{\{s,a\}})$ by μ. Consider then the decoration of the above words:

$$\begin{array}{rcl} \delta([w_1a_1 \dots w_{p+1}]^{\{s,a\}}) &=& \delta([u]^{\{s,a\}})(s,q_{\mu+1})\dots(s,q_{\mu+\overline{\eta}})(s,q_{\mu+\overline{\eta}+1})\dots(s,q_{\mu+\eta}), \\ \delta([\overline{w}_1a_1 \dots \overline{w}_{p+1}]^{\{s,a\}}) &=& \delta([u]^{\{s,a\}})(s,q_{\mu+1})\dots(s,q_{\mu+\overline{\eta}}). \end{array}$$

Using the notations

$$\begin{array}{rcl} W &=& \delta([w_1a_1 \dots w_{p+1}]^{\{s,a\}}), \\ \overline{W} &=& \delta([\overline{w}_1a_1 \dots \overline{w}_{p+1}]^{\{s,a\}}), \\ V &=& (s,q_{\mu+\overline{\eta}+1})\dots(s,q_{\mu+\eta}), \end{array}$$

the relation between W and $\overline{W}$ can be formulated in a short way:

$$W = \overline{W}V. \tag{4.9}$$

Reconstruction of V: By Lemma 3.17 we can reconstruct k last letters of the word W or the whole word W from the a-diary $\boldsymbol{E}_{\boldsymbol{\lambda}}(\gamma)$. Here:

$$k = (\kappa-1)r + 1 - \ell([a_{p+1}w_{p+2}\dots w_{p+r}a_{p+r}]^{\{s,a\}}). \tag{4.10}$$

We show that $k > \ell(V)$. Then V can be reconstructed from the diary.

$$\begin{array}{rcl} (\kappa-1)r+1 & \overset{\kappa>3n}{\geq} & 3n\cdot r+1 \\ & \overset{r\geq\frac{c}{3n}}{\geq} & c+1 \\ & > & c \\ & \overset{c=d(\gamma,\overline{\gamma})}{=} & \ell(v)+\ell(\overline{v}) \\ & \geq & \ell(v) \\ & \geq & \ell([v]^{\{s,a\}}) \\ & \overset{\text{Rem. } 3.7}{\geq} & \underbrace{\ell([v_{p+1}]^{\{s,a\}})}_{\eta} + \ell([a_{p+1}v_{p+2}\dots v_{p+r}a_{p+r}]^{\{s,a\}}) \\ & \geq & \underbrace{\ell(V)}_{\eta-\overline{\eta}} + \ell([a_{p+1}v_{p+2}\dots v_{p+r}a_{p+r}]^{\{s,a\}}). \end{array}$$

Because all the letters s from u_{p+1} are in w_{p+1} by (4.8), they can not be in w_{p+i}, $i > 1$. Thus it holds:

$$\ell([a_{p+1}w_{p+2}\dots w_{p+r}a_{p+r}]^{\{s,a\}}) = \ell([a_{p+1}v_{p+2}\dots v_{p+r}a_{p+r}]^{\{s,a\}}).$$

This implies:

$$(\kappa - 1)r + 1 > \ell(V) + \ell([a_{p+1}w_{p+2}\dots w_{p+r}a_{p+r}]^{\{s,a\}}). \tag{4.11}$$

By (4.10) and (4.11) it follows:

$$k > \ell(V).$$

Thus V can be reconstructed from the a-diary up to entry $p + r$. Because these diaries of γ and $\overline{\gamma}$ coincide (4.7), we also can reconstruct the last $\ell(V)$ letters of $\overline{W}$ and conclude that these are as well $V = (s, q_{\mu+\overline{\eta}+1})\dots(s, q_{\mu+\eta})$. Thus $\overline{W}$ ends with V, i.e.:

$$\overline{W} = \overline{V}V,$$

where $\overline{V}$ is the beginning of $\overline{W}$ with the appropriate length.

By combining the statement above with (4.9) it follows:

$$\begin{aligned} W &= \overline{W}V \\ &= \overline{V}VV \\ &= \overline{V}(s, q_{\mu+\overline{\eta}+1})\dots(s, q_{\mu+\eta})(s, q_{\mu+\overline{\eta}+1})\dots(s, q_{\mu+\eta}). \end{aligned}$$

By the assumption of Case 2.1 (i), we know that $\eta - \overline{\eta} > 0$. Thus by definition of the decoration map δ, the sequence

$$q_{\mu+\overline{\eta}+1}\dots q_{\mu+\eta}q_{\mu+\overline{\eta}+1}\dots q_{\mu+\eta} = (q_{\mu+\overline{\eta}+1}\dots q_{\mu+\eta})^2$$

is a nonempty (square) subsequence of the square free sequence $(q_i)_{i\in\mathbb{N}}$, which is a contradiction.

Case 2.2: The words v_{p+1} and $\overline{v}_{p+1}$ have the same number of letters $s = x_i$ and $t = \overline{x}_i$:

$$\begin{aligned} (i) &\quad \ell_s(v_{p+1}) = \ell_s(\overline{v}_{p+1}) \quad \text{and} \\ (ii) &\quad \ell_t(v_{p+1}) = \ell_t(\overline{v}_{p+1}). \end{aligned}$$

Then:

$$v_{p+1} = x_\ell \dots s x_{i-1}\dots x_1 \quad \text{and} \quad \overline{v}_{p+1} = \overline{x}_{\overline{\ell}}\dots t x_{i-1}\dots x_1.$$

The assumption of this case implies that there are $j \in \{i+1, \dots, \ell\}$ and $k \in \{i+1, \dots, \overline{\ell}\}$ with $x_j = t$ and $\overline{x}_k = s$, thus we have:

$$v_{p+1} = x_\ell \dots t \dots s x_{i-1} \dots x_1 \quad \text{and} \quad \overline{v}_{p+1} = \overline{x}_\ell \dots s \dots t x_{i-1} \dots x_1.$$

By Lemma 3.9 (3) the ending of $[v_{p+1}]^{\{s,t\}}$ remains unchanged by the normalization of uv:

$$[w_{p+1}]^{\{s,t\}} = \dots s[x_{i-1} \dots x_1]^{\{s,t\}}.$$

By Remark 3.7 it follows:

$$[w_1 a_1 \dots w_{p+1}]^{\{s,t\}} = \dots s[x_{i-1} \dots x_1]^{\{s,t\}}.$$

By an analogous argumentation for the word $\overline{w}$ it holds:

$$[\overline{w}_1 \overline{a}_1 \dots \overline{w}_{p+1}]^{\{s,t\}} = \dots t[x_{i-1} \dots x_1]^{\{s,t\}}.$$

Denote $s[x_{i-1} \dots x_1]^{\{s,t\}}$ by V. Then V can be reconstructed by Lemma 3.17 as in Case 2.1 from the a-diary $\boldsymbol{E_\lambda}(\gamma)$. In the same way $V' = t[x_{i-1} \dots x_1]^{\{s,t\}}$ can be reconstructed from the entries $\overline{\omega}_{p+1} \dots \overline{\omega}_{p+r}$ of the a-diary $\boldsymbol{E_\lambda}(\overline{\gamma})$.

The diary entries coincide by (4.7), but the reconstructed words differ

$$V = s[x_{i-1} \dots x_1]^{\{s,t\}} \neq t[x_{i-1} \dots x_1]^{\{s,t\}} = V'.$$

This is a contradiction. Hence the proof is complete. $\square$

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